BCS-012 BASIC MATHEMATICS

Block-2, Unit-2 Complex Numbers

No

Date 04/07/2021

(iota) $i = \sqrt{-1}$

Natural numbers (N)
Integers (Z)
rational numbers (Q)
real numbers (R).

Now, we have to study 'Complex Numbers (C)'.

- Italian Mathematician 'Cardano' was puzzled by the number $\sqrt{-15}$ during his research and could not define it.
- Finally, Carl Gauss, the greatest mathematician correctly formulate the concept of complex numbers.
- In this chapter, complex numbers, Block-2, Unit-2, we will study the (i) different ways of representing complex numbers,
  (ii) basic algebraic operations on them.
  (iii) De Moivre' Theorem on complex numbers.

Swiss mathematician Euler invent the term 'imaginary unit' in (1777 sm) for $\sqrt{-1}$. He denoted it by the Greek letter iota, that is i.
He put $\boxed{i = \sqrt{-1}}$

German Mathematician Carl Gauss started calling numbers like $1+\sqrt{-1}$ $(1+i)$ or $-5+i\sqrt{7}$, $(-5+\sqrt{-7})$ as 'Complex numbers'.

Date 04/07/2021

→ $x + iy$ is a Complex number

→ Here, x and y are real numbers,
and $i^2 = -1$

⇒ In $x + iy$, x is called as the real part
and is called the imaginary part.

→ x is denoted as $x = Re(x + iy)$ and
y is denoted as $y = Im(x + iy)$

noted:

- i is not a real number
- $Im(x + iy)$ indicates the real number y and not iy

All set of Complex numbers is denoted as C

Example: $C = \{x + iy \mid x, y \in R\}$

However, the complex number
$x + iy$ normally written as Z
i.e, mean. $Z = x + iy = Re Z + i Im Z$

However, when $y \in Q$ [∵ Q is rational number]
we generally $x + yi$.

Example: $5 + \frac{7}{9}i$ not $5 + i\frac{7}{9}$.

⇒ Purely Real and Purely Imaginary numbers.

If $y = 0$, Z is called purely real.

If $x = 0$, Z is called purely imaginary.

⇒ Complex Conjugate

Let $Z = x + iy \in C$

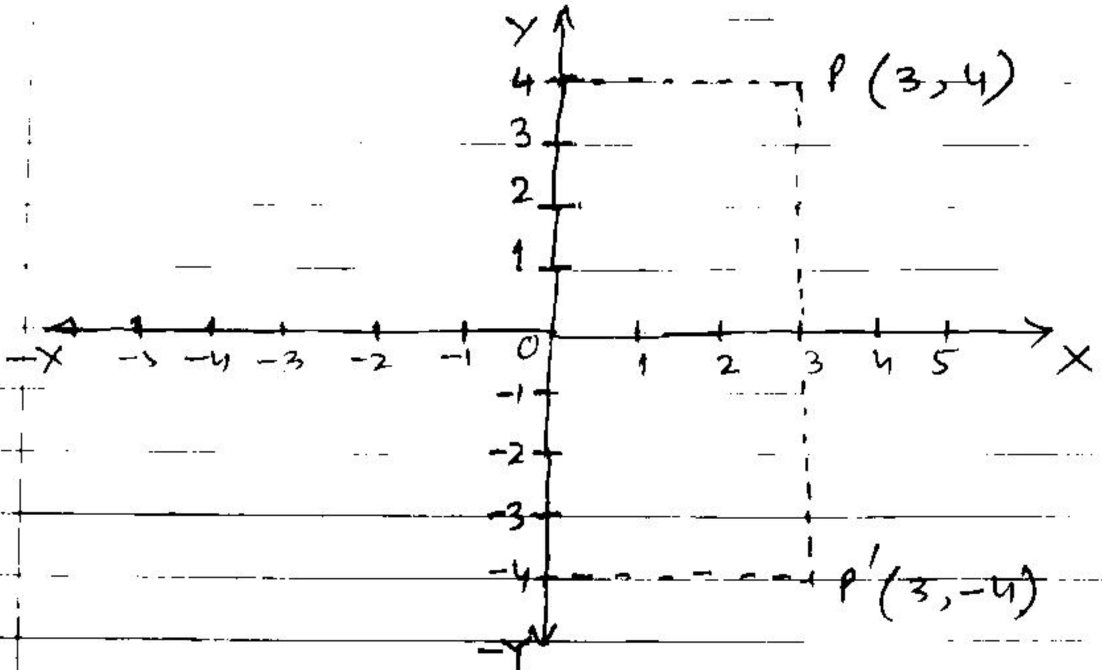
Now The complex conjugate or Conjugate of Z is defined

$\bar{Z} = x - iy$

$Re \bar{Z} = Re Z$ and $Im \bar{Z} = -Im Z$.

SPIRX

⇒ Argand Diagram:- Complex No., $3 - 4i$.



The point $P'(3,-4)$ is represents the conjugate at P i.e. $\overline{z} = 3 - 4i$.

Purely real numbers lie along the x-axis, and purely imaginary numbers lie along the y-axis.

Addition in C

C.Ns $= z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

$c = z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)$ that is

$(x_1 \cdot y_1) + (x_2 \cdot y_2) = (x_1 + x_2 \cdot y_1 + y_2)$

Subtraction in C

difference $z_1 - z_2$

C.Ns. $z_1 = x_1 + iy_2$ and $z_2 = x_2 + iy_2$ is

$z_1 + (-z_2)$,

Here, $\cancel{-z_2} = (-x_2) + i(-y_2)$.

Hence, $z_1 - z_2 = \cancel{z_1 + (-z_2)}$

$= (x_1 + iy_1) + [(-x_2) + i(-y_2)]$

$= (x_1 - x_2) + i(y_1 - y_2)$

P.T.O. (4)

Complex Nos.

SPIRX

⇒ Example of Additive identity in C. ~~of (x+iy)~~

Value of $z - z$

$\therefore z - z = (x+iy) - (x+iy)$

$= (x-x) + i(y-y)$

$= 0 + i0 = 0$

⇒ multiplication of Complex Nos :- $z_1 z_2$

$z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$

⇒ Let z_1 and z_2 be two given C. Ns such that in polar form

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

then,

$z_1 \cdot z_2 = r_1 \cdot r_2 (\cos\theta_1 + i\sin\theta_1) \cdot (\cos\theta_2 + i\sin\theta_2)$

$= r_1 \cdot r_2 \{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\}$

$= r_1 r_2 (\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$

$\because |z_1 z_2| = r_1 r_2 = |z_1| \cdot |z_2|$.

and $Arg(z_1 z_2) = (\theta_1 + \theta_2) + 2K\pi$

where K is constant $K \in \mathbb{Z}$

hence, $-\pi < \theta_1 + \theta_2) + 2K\pi \le \pi$

[The value of θ is Arg = Argument Arg z]

⇒ Std. Trigonometric relations

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\sin(A+B) = \sin A\cos B + \cos A\sin B$.

B2-U2 Complex Nos.

No.
Date: 04/07/2021

SPIRX

⇒ Division of Complex Nos -

Example. $\frac{5}{3-\sqrt{2}}$

Now,
multiplied numerator and denominator by the conjugate of the denominator, $3+\sqrt{2}$:

$$\frac{5}{3-\sqrt{2}} \times \frac{3+\sqrt{2}}{3+\sqrt{2}} = \frac{15+5\sqrt{2}}{9-2}$$

$$= \frac{15-5\sqrt{2}}{7}.$$

$\left[\sqrt{2}\cdot\sqrt{2} = \sqrt{2\cdot 2} = \sqrt{2^2} = \cancel{2}^{\frac{1}{\cancel{2}}\times 2} = 2 \right.$

⇒ Example :

Express $\frac{3-j}{4-2j}$ in the form $x+yi$.

Sol: The conjugate of $4-2j$ is $4+2j$.
Then, multiply the top and bottom of the fraction by this conjugate, i.e. $4+2j$.

$$\frac{3-i}{4-2i} \times \frac{4+2i}{4+2i} = \frac{12+6i-4i-2i^2}{16-4i^2}$$

$$= \frac{12+2+6i-4j}{16+4} = \frac{14+2i}{20} = \frac{7+i}{10}.$$

Ans

⇒ Simplify: $\frac{1-\sqrt{-4}}{2+9i}$

Soln $\frac{1-\sqrt{-4}}{2+9i} = \frac{1-2i}{2+9i} \times \frac{2-9i}{2-9i}$

$$= \frac{2-9i-4i+18i^2}{4-81i^2}$$

$$= \frac{2-13i+18\times -1}{\cancel{4+81}}$$

$$= \frac{-16-13i}{4+81} = \frac{-16-13i}{85}$$

Ans

$\left[\because i = \sqrt{-1} \right.$
$\sqrt{-4} = \sqrt{-2^2}$
$= \sqrt{-1}\cdot 2$
$= 2i \left. \right]$

$\left[\because i^2 = -1 \right]$

$\left[\frac{4-81(-1)}{4+81} \right]$

SPIRX

⇒ Polar form of Z.

$x = r\cos\theta \quad , y = r\sin\theta$

$\therefore Z = x + iy$

$= r\cos\theta + i\sin\theta$

$= r(\cos\theta + i\sin\theta)$

Where, $r = |Z|$ and $\theta = \operatorname{Arg} Z$.

This is called polar form of Z.

From above, when $Z = x + iy$,

$$\theta = \operatorname{Arg} Z = \tan^{-1}\left(\frac{y}{x}\right).$$

⇒ DE MOIVRE's THEOREM :-

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, for any $n \in \mathbb{Z}$ and any angle θ.

that states.

If $Z = r(\cos\theta + i\sin\theta)$ then

$Z^n = r^n(\cos n\theta + i\sin n\theta)$ for any integer n.

Q Example.

If $(\cos\theta + i\sin\theta)^2 = x + iy$, that show $x^2 + y^2 = 1$

Sol Given that

$(\cos\theta + i\sin\theta)^2 = x + iy$

$|\cos\theta + i\sin\theta|^2 = |x + iy|$

$|\cos\theta + i\sin\theta|^2 = \sqrt{x^2 + y^2}$

$\left(\sqrt{\cos^2\theta + \sin^2\theta}\right)^2 = \sqrt{x^2 + y^2}$

Therefore, $x^2 + y^2 = 1$.